8-th Japanese Mathematical Olympiad 1998

Final Round – February

1. Let $p \geq 3$ be a prime number and let $A_0, A_1, \dots, A_{p-1}$ be points on a circle in this order. For each $k = 1, 2, \dots, p$ we write the number k at point $A_{1+\cdots+(k-1)}$. How many points have at least one number written at them?

2. In a country there are 1998 airports connected by some direct flights. Among any three airports, some two are not connected by a direct flight. What is the maximum possible number of direct flight?

3. Let $P_1P_2 \dots P_n$ be a closed polygonal line. The external angle at P_i is defined as $180^\circ - \alpha_i$, where α_i is the oriented angle between the rays P_iP_{i-1} and P_iP_{i+1} taken in the range $(0^\circ, 360^\circ)$ (here $P_0 = P_n$ and $P_{n+1} = P_1$). Prove that if the sum of the external angles is a multiple of 720°, then the number of self-intersections of the polygonal line is odd.

4. Let $c_{n,m}$ be the number of permutations of $\{1, 2, \dots, n\}$ which can be written as the composition of m transpositions of the form $(i, i+1)$ ($i \in \{1, \dots, n-1\}$) but not of $m-1$ such transpositions. Prove that for all $n \in \mathbb{N}$,

$$\sum_{m=0}^{\infty} c_{n,m} t^m = \prod_{i=1}^{n} (1 + t + \cdots + t^{i-1}).$$

5. A marker with one white side and one black side is put on each of 12 points around a circle. A legal move consists of selecting a black marker and reversing its two neighbors. Find all initial configurations which can be reduced to a configuration with all markers but one white in finitely many legal moves.

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