Japanese Mathematical Olympiad 1994

Final Round – February

1. For any positive integer n, let a_n denote the closest integer to $\sqrt{n}$, and let $b_n = n + a_n$. Determine the increasing sequence (c_n) of positive integers which do not occur in the sequence (b_n).

2. Five points, no three collinear, are given on the plane. Let $l_1, l_2, \dots, l_{10}$ be the lengths of the ten segments joining any two of the given points. Prove that if $l_1^2, \dots, l_9^2$ are rational numbers, then l_{10}^2 is also a rational number.

3. Let P_0 be a point in the plane of triangle $A_0A_1A_2$. Define P_i $(i = 1, \dots, 6)$ inductively as the point symmetric to P_{i-1} with respect to A_k, where k is the remainder when i is divided by 3.

   (a) Prove that $P_6 \equiv P_0$.

   (b) Find the locus of points P_0 for which P_iP_{i+1} does not meet the interior of $\triangle A_0A_1A_2$ for $0 \leq i \leq 5$.

4. In a triangle ABC, M is the midpoint of BC. Given that $\angle MAC = 15^\circ$, find the maximum value of $\angle ABC$.

5. In a deck of N cards, the cards are denoted by 1 to N. These cards are dealt to N people twice. A person X wins a prize if there is no person Y who got a card with a smaller number than X both times. Determine the expected number of prize winners.

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