1-st Japanese Mathematical Olympiad 1991

Final Round – February 15

1. Let P, Q and R be points on the sides BC, CA and AB of a triangle ABC respectively, such that $\overrightarrow{BP} : \overrightarrow{PC} = \overrightarrow{CQ} : \overrightarrow{QA} = \overrightarrow{AR} : \overrightarrow{RB} = t : (1-t)$ for some real number t. Prove that there is a triangle Δ whose side lengths are AP, BQ, CR, and find the ratio of the area of triengle ABC to that of Δ in terms of t.

2. Let p and q be mappings from $\mathbb{N}$ to itself given by

$$\begin{array}{lllll} p(1)=2, & p(2)=3, & p(3)=4, & p(4)=1, & p(n)=n \text{ for } n \geq 5; \\ q(1)=3, & q(2)=4, & q(3)=2, & q(4)=1, & q(n)=n \text{ for } n \geq 5. \end{array}$$

   (a) Find a mapping $f \colon \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = p(n)+2$ for $n \geq 1$.

   (b) Prove that there is no mapping $g \colon \mathbb{N} \to \mathbb{N}$ such that $g(g(n)) = q(n)+2$ for $n \geq 1$.

3. Let A be a positive 16-digit integer. Show that we can find some consecutive digits of A whose product is a perfect square.

4. Let be given a 10×14 matrix (a_{ij}) with each a_{ij} being equal to 0 or 1, such that each column or row contains an odd number of ones. Prove that among the a_{ij} with an even $i+j$ there are an even number of ones.

5. A set S of distinct $n \geq 2$ points is given on a plane. Show that there are two distinct points $P_i, P_j \in S$ such that the circle with diameter P_iP_j contains at least $[n/3]$ of the points from S.