Japanese IMO Team Selection Test 1990

Final Round – February 11

1. Nonempty subsets A_1, A_2, A_3, A_4 and A_5 of $\mathbb{R}^3$ are such that:

   (i) $A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 = \mathbb{R}^3$;

   (ii) $A_i \cap A_j = \emptyset$ for $i \neq j$.

   Prove that there exists a plane which intersects at least four of the A_i's.

2. Let $1 \leq a_0 < a_1 < \cdots < a_n \leq 2n-3$ be integers, where $n \geq 3$ is an integer. Prove that there exist different indices i, j, k, l, m such that $a_i + a_j = a_k + a_l = a_m$.

3. A nonempty set X of positive integers has the property that, for any $x \in X$, numbers $4x$ and $[\sqrt{x}]$ are also in X. Prove that $X = \mathbb{N}$.

4. Let $n > 2$ be an integer. Find the maximum K and the minimum G such that for any positive numbers $a_1, a_2, \ldots, a_n$ the following inequality holds:

$$K < \frac{a_1}{a_1 + a_2} + \frac{a_2}{a_2 + a_3} + \cdots + \frac{a_n}{a_n + a_1} < G.$$

5. Consider the set $Q(n)$ of all words of size $2n$ consisting of n letters A and n letters B which have the following property: For any $k \leq 2n$, among the first k letters of the word there are at least as many letters B as there are letters A. Determine the cardinalities of (a) $Q(8)$ and (b) $Q(n)$ for any n.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com