19-th Japanese Mathematical Olympiad 2009

Final Round

1. Find all positive integers n such that $2^n + n | 8^n + n$.

2. Let N be a positive integer. A sequence of integers from the set $\{1,2,\dots,N\}$ is chosen in such a way that each element of $\{1,2,\dots,N\}$ appears in the sequence, and the sum of all elements is even. Prove that it is possible to paint each of the chosen numbers in red or green (each number with one color) so that the sum of the red numbers is equal to the sum of the green numbers.

3. Let $k \geq 2$ be an integer. Let n_1, n_2, n_3 be positive integers and a_1, a_2, a_3 some integers from the set $\{1,2,\dots,k-1\}$. If $b_i = a_i \cdot \sum_{j=0}^{n_i} k^j$, $(i = 1,2,3)$, find all possible triples of integers (n_1,n_2,n_3) such that $b_1 b_2 = b_3$.

4. Let Γ be the circumcircle of $\triangle ABC$. A circle with center O touches the segment BC at P and the arc BC of Γ not containing A at Q. If $\angle BAO = \angle CAO$, prove that $\angle PAO = \angle QAO$.

5. Find all functions $f : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}_+ \cup \{0\}$ such that

$$f(x^2) + f(y) = f(x^2 + y + x f(4y))$$

for all non-negative real numbers x and y.

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