18-th Japanese Mathematical Olympiad 2008

Final Round, February 11

1. Given a polynomial $P(x)$ with integer coefficients, assume that $P(n^2) = 0$ for some non-zero integer n. Prove that $P(a^2) \neq 1$ for all rational numbers $a \neq 0$.

2. There are 2008 red and 2008 white cards. 2008 players are sitting around a circle. Each player is given 2 cards of the same color. In each of the turns, every player does one of the following two things:

   (i) If the player has one or more red cards, he will pass one red card to the player immediately to his left;

   (ii) If the player has no red cards, he will pass one (white) card to his left neighbor.

   Find the maximal number of turns required for each player to have one red and one white card.

3. Let O be the circumcenter of the acute-angled triangle ABC. A circle passing through A and O intersects the lines AB and AC at P and Q respectively. If the lengths of the segments PQ and BC are equal, find the length of the angle between the lines PQ and BC.

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x+y)f(f(x)-y) = xf(x) - yf(y)$ for all $x, y, \in \mathbb{R}$.

5. Does there exist a natural number n such that:

   For every rational number a there are integer b and non-zero integers $a_1, \dots, a_n$ such that $a = b + \sum_{i=1}^{n} \frac{1}{a_i}$?

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