17-th Japanese Mathematical Olympiad 2007

Final Round – February 11

1. Let n be a positive integer. Two players alternately choose numbers from the set $\{1, 2, \ldots, n\}$ until the set is exhausted. At the end, the first player wins if the sum of the numbers he chosen is divisible by 3; otherwise the second player wins. For which n does the first player have a winning strategy?

2. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}$ such that for all $x, y > 0$

$$f(x) + f(y) \le \frac{f(x+y)}{2}, \quad \frac{f(x)}{x} + \frac{f(y)}{y} \ge \frac{f(x+y)}{x+y}.$$

3. Let Γ be the circumcircle of triangle ABC. Denote the circle tangent to AB, AC and internally to Γ by Γ_A. Define Γ_B and Γ_C analogously. Let $\Gamma_A, \Gamma_B, \Gamma_C$ touch Γ at P, Q, R, respectively. Prove that the lines AP, BQ and CR are concurrent.

4. A *band* of width d is the set of points in the plane that are on the distance at most $d/2$ from a line. Suppose that any three of four given points in the plane can be covered by a band of width 1. Show that all four points can be covered by a band of width $\sqrt{2}$.

5. For each positive number x define $A(x) = \{[nx] \mid n \in \mathbb{N}\}$. Find all irrational numbers $\alpha > 1$ with the following property: If a positive number β is such that $A(\alpha) \supset A(\beta)$, then β/α is an integer.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com