16-th Japanese Mathematical Olympiad 2006

Final Round – February 11

1. Five distinct points A, M, B, C, D are on a circle o in this order with $MA = MB$. The lines AC and MD meet at P and the lines BD and MC meet at Q. If the line PQ meets the circle o at X and Y, prove that $MX = MY$.

2. Find all integers k for which there exist infinitely many triples (a,b,c) of integers satisfying $(a^2-k)(b^2-k) = c^2-k$.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any x, y
$$f(x)^2 + 2yf(x) + f(y) = f(y + f(x)).$$

4. Let m, n, a, a', b, b' be positive integers with $2 \le m \le n$, $a \le m$, $a' \le m$, $b \le n$, $b' \le n$, and $(a,b) \ne (a',b')$.

   There is a town in the shape of a rectangular grid with m avenues and n streets perpendicular to avenues. The x-th avenue from the west and the y-th street from the north intersect at (x,y). For which m,n,a,b,a',b' does there exist a path from (a,b) to (a',b') crossing all the intersections exactly once?

5. Find the maximum value of A such that for any positive numbers x_i, y_i, z_i ($i = 1,2,3$) the following inequality holds:
$$(x_1^3+x_2^3+x_3^3+1)(y_1^3+y_2^3+y_3^3+1)(z_1^3+z_2^3+z_3^3+1) \\ \ge A(x_1+y_1+z_1)(x_2+y_2+z_2)(x_3+y_3+z_3).$$

   For this maximum value of A find the cases of equality.

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