15-th Japanese Mathematical Olympiad 2005

Final Round – February 11

1. We are given a 17×17 array of coins with heads up. In each step one can choose five consecutive coins in a row, column or diagonal and reverse them. Is it possible to obtain a position with all coins having tails up in a finite number of such steps?

2. Let $P(x,y)$ and $Q(x,y)$ be polynomials with integer coefficients. Given integers a_0, b_0, define the sequence of points $X_n(a_n, b_n)_{n \geq 0}$ by $a_{n+1} = P(a_n, b_n)$ and $b_{n+1} = Q(a_n, b_n)$. Suppose that $X_1 \neq X_0$, but $X_k = X_0$ for some $k \in \mathbb{N}$. Show that the number of lattice points on the segment $X_n X_{n+1}$ is the same for each n.

3. If a, b, c are positive numbers with $a+b+c=1$, prove the inequality
$$a\sqrt[3]{1+b-c} + b\sqrt[3]{1+c-a} + c\sqrt[3]{1+a-b} \leq 1.$$

4. The tangents to a circle Γ from a point X meet the circle at points A and B. A line through X intersects the circle at C and D with D between X and C so that the lines AC and BD are perpendicular and meet at F. Let CD meet AB at G and let the perpendicular bisector of GX meet the segment BD at H. Prove that the points X, F, G, and H lie on a circle.

5. The boss has to assign ten job positions to ten candidates, considering two parameters: preference and ability. If candidate A prefers job v to job u and has a better ability in job v than candidate B, but A is assigned job u and B is assigned job v, then A will complain. Also, if it is possible to assign each job to a candidate with a higher ability, the director will complain. Show that the boss can assign the jobs so as to avoid any complaints.

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