13-th Japanese Mathematical Olympiad 2003

Final Round – February 11

1. A point P lies in a triangle ABC. The lines BP and CP meet AC and AB at Q and R respectively. Given that $AR = RB = CP$ and $CQ = PQ$, find $\angle BRC$.

2. We have two distinct positive integers a and b with $a \mid b$. Each of a and b consists of $2n$ decimal digits with the leftmost digit nonzero. Furthermore, the first n digits of a are identical to the last n digits of b and vice versa, as in $n = 2$, $a = 1234$, $b = 3412$ (although this example does not satisfy $a \mid b$). Determine a and b.

3. Find the greatest real number k such that, for any positive a,b,c with $a^2 > bc$,
$$(a^2 - bc)^2 > k(b^2 - ca)(c^2 - ab).$$

4. Let p and $q \geq 2$ be coprime integers. A list of integers $(r, a_1, a_2, \dots, a_n)$ with $|a_i| \geq 2$ for all i is said to be an *expansion* of p/q if
$$\frac{p}{q} = r + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}.$$
For example, $(-1, -3, 2, -2)$ is an expansion of $\frac{-10}{7}$. Now define the *weight* of an expansion $(r, a_1, a_2, \dots, a_n)$ to be the product
$$(|a_1| - 1)(|a_2| - 2)\cdots(|a_n| - 1).$$
Show that the sum of the weights of all expansions of p/q is q.

5. Find the greatest possible integer n such that one can place n points in a plane with no three on a line, and color each of them either red, green, or yellow so that:
   - (i) inside each triangle with all vertices red there is a green point;
   - (ii) inside each triangle with all vertices green there is a yellow point;
   - (iii) inside each triangle with all vertices yellow there is a red point.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com