

12-th Japanese Mathematical Olympiad 2002

Final Round – February 11

1. Distinct points A, M, B with $AM = MB$ are given on a circle C_0 . Let P be a point on the arc AB not containing M . Circle C_1 is internally tangent to C_0 at P and tangent to AB at Q . Prove that the product $MP \cdot MQ$ is independent of the position of P .
2. There are $n \geq 3$ coins on a circle. Consider a coin and the two coins adjacent to it; if there are an odd number of heads among the three, we call it *good*. An operation consists of turning over all good coins simultaneously. Initially, exactly one of the n coins is a head. The operation is repeatedly performed.
 - (a) Prove that if n is odd, the coins will never be all-tails.
 - (b) For which values of n is it possible to make the coins all-tails after several operations? Find, in terms of n , the number of operations needed for this to occur.
3. Denote by $S(n)$ the sum of decimal digits of a positive integer n . Show that there exist 2002 distinct positive integers n_1, \dots, n_{2002} such that

$$n_1 + S(n_1) = n_2 + S(n_2) = \dots = n_{2002} + S(n_{2002}).$$

4. For any positive numbers $a_1, \dots, a_n, b_1, \dots, b_n$ ($n \geq 3$) satisfying $a_1 + a_2 + \dots + a_n = 1$ and $b_1^2 + b_2^2 + \dots + b_n^2 = 1$, prove the inequality

$$a_1(b_1 + a_2) + a_2(b_2 + a_3) + \dots + a_n(b_n + a_1) < 1.$$

5. Let S be a set of 2002 points in the coordinate plane, no two of which have the same x - or y -coordinate. For any two points $P, Q \in S$, consider the rectangle with one diagonal PQ and the sides parallel to the axes. Denote by W_{PQ} the number of points of S lying in the interior of this rectangle. Determine the maximum N such that, no matter how the points of S are distributed, there always exist points P, Q in S with $W_{PQ} \geq N$.