Italian IMO Team Selection Test 1999

Cortona, May 29, 1999

Time allowed: 4 hours

1. Prove that for any prime number p the equation $2^p + 3^p = a^n$ has no solution (a,n) in integers greater than 1.

2. Let D and E be points on sides AB and AC respectively of a triangle ABC such that DE is parallel to AB and tangent to the incircle of ABC. Prove that

$$DE \leq \frac{1}{8}(AB+BC+CA).$$

3. (a) Find all strictly monotone functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+f(y)) = f(x)+y \qquad \text{for all real } x,y.$$

   (b) If $n > 1$ is an integer, prove that there is no strictly monotone function $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x+f(y)) = f(x)+y^n \qquad \text{for all real } x,y.$$

4. Let X be an n-element set and let $A_1,\dots,A_m$ be subsets of X such that

   (i) $|A_i| = 3$ for each $i = 1,\dots,m$;

   (ii) $|A_i \cap A_j| \leq 1$ for any two distinct indices i, j.

   Show that there exists a subset of X with at least $\left[\sqrt{2n}\right]$ elements which does not contain any of the A_i's.