Italian IMO Team Selection Test 2003

First Day – Pisa, May

1. Find all triples (a,b,p) with a,b positive integers and p a prime number such that $2^a + p^b = 19^a$.

2. Let $B \neq A$ be a point on the tangent to circle S_1 through point A on the circle. A point C outside the circle is chosen so that segment AC intersects the circle in two distinct points. Let S_2 be the circle tangent to AC at C and to S_1 at some point D, where D and B are on the opposite sides of the line AC. Let O be the circumcenter of triangle BCD. Show that O lies on the circumcircle of triangle ABC.

3. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(f(x)+y) = 2x + f(f(y)-x) \quad \text{for all real } x,y.$$

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4. The incircle of a triangle ABC touches the sides AB,BC,CA at points D,E,F, respectively. The line through A parallel to DF meets the line through C parallel to EF at G.

   (a) Prove that the quadrilateral $AICG$ is cyclic.

   (b) Prove that the points B,I,G are collinear.

5. For n an odd positive integer, the unit squares of an $n \times n$ chessboard are colored alternately black and white, with the four corners colored black. A *tromino* is an L-shape formed by three connected unit squares.

   (a) For which values of n is it possible to cover all the black squares with nonoverlapping trominos lying entirely on the chessboard?

   (b) When it is possible, find the minimum number of trominos needed.

6. Let $p(x)$ be a polynomial with integer coefficients and let n be an integer. Suppose that there is a positive integer k for which $f^{(k)}(n) = n$, where $f^{(k)}(x)$ is the polynomial obtained as the composition of k polynomials f. Prove that $p(p(n)) = n$.