Italian IMO Team Selection Test 2002

First Day – Cortona, May

1. Given that in a triangle ABC, $AB = 3$, $BC = 4$ and the midpoints of the altitudes of the triangle are collinear, find all possible values of the length of AC

2. Prove that for each prime number o and positive integer n, p^n divides
$$\binom{p^n}{p} - p^{n-1}.$$

3. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the following conditions:

   (i) $f(x+yf(x)) = f(x)f(y)$ for all $x,y > 0$;

   (ii) there are at most finitely many x with $f(x) = 1$.

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4. A scalene triangle ABC is inscribed in a circle Γ. The bisector of angle A meets BC at E. Let M be the midpoint of the arc BAC. The line ME intersects Γ again at D. Show that the circumcenter of triangle AED coincides with the intersection point of the tangent to Γ at D and the line BC.

5. On a soccer tournament with $n \geq 3$ teams taking part, several matches are played in such a way that among any three teams, some two play a match.

   (a) If $n = 7$, find the smallest number of matches that must be played.

   (b) Find the smallest number of matches in terms of n.

6. Prove that for any positive integer m there exist an infinite number of pairs of integers (x,y) such that (i) x and y are relatively prime; (ii) x divides y^2+m; (iii) y divides x^2+m.

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