4-th Italian Mathematical Olympiad 1988

Viareggio, April 25, 1988

1. Players A and B play the following game: A tosses a coin n times, and B does $n+1$ times. The player who obtains more ”heads” wins; or in the case of equal balances, A is assigned victory. Find the values of n for which this game is fair (i.e. both players have equal chances for victory).

2. In a basketball tournament any two of the n teams $S_1, S_2, \dots, S_n$ play one match (no draws). Denote by v_i and p_i the number of victories and defeats of team S_i ($i = 1, 2, \dots, n$), respectively. Prove that
$$v_1^2 + v_2^2 + \cdots + v_n^2 = p_1^2 + p_2^2 + \cdots + p_n^2.$$

3. A regular pentagon of side length 1 is given. Determine the smallest r for which the pentagon can be covered by five discs of radius r and justify your answer.

4. Show that all terms of the sequence $1, 11, 111, 1111, \dots$ in base 9 are triangular numbers, i.e. of the form $\frac{m(m+1)}{2}$ for an integer m.

5. Given four non-coplanar points, is it always possible to find a plane such that the orthogonal projections of the points onto the plane are the vertices of a parallelogram? How many such planes are there in general?

6. The edge lengths of the base of a tetrahedron are a, b, c, and the lateral edge lengths are x, y, z. If d is the distance from the top vertex to the centroid of the base, prove that
$$x + y + z \leq a + b + c + 3d.$$

7. Given $n \geq 3$ positive integers not exceeding 100, let d be their greatest common divisor. Show that there exist three of these numbers whose greatest common divisor is also equal to d.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com