22-nd Italian Mathematical Olympiad 2006

Cesenatico, May 5, 2006

1. Rosa and Savino are playing the following game with Napolitan playing cards (40 cards numbered 1 through 10 with four different signs). Initially each player gets 20 cards. The players alternate putting cards (one per move) onto the table. A player after whose move there are several cards on the table whose values sum up to 15, removes these cards. It turned out at the end of the game that Savino kept two cards with values 5 and 3, Rosa kept one card, and a card with value 9 remained on the table. What is the value of the Rosa's card?

2. Find all triples (m,n,p) such that $p^n + 144 = m^2$, where m and n are positive integers and p a prime number.

3. Let A and B be distinct points on a circle Γ that are not diametrically opposite. Point P is different from A and B and varies on Γ. Find the locus of the orthocenter of triangle ABP.

4. The squares of the infinite chessboard are numbered $1,2,\dots$ along a spiral, as shown in the picture. A *right ray* is a sequence of squares obtained by starting at one square and going to the right.

	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\vdots$
17	16	15	14	13	$\vdots$
18	5	4	3	12	$\vdots$
19	6	1	2	11	$\vdots$
20	7	8	9	10	$\vdots$
21	22	23	24	25	$\cdots$

   (a) Show that there is a right ray not containing a multiple of 3.

   (b) Prove that there are infinitely many pairwise disjoint right rays which do not contain multiples of 3.

5. Consider the inequality
$$(x_1 + x_2 + \cdots + x_n)^2 \geq 4(x_1x_2 + x_2x_3 + \cdots + x_nx_1).$$

   (a) For which $n \geq 3$ is this inequality true for all positive x_i?

   (b) For which $n \geq 3$ is it true for all real numbers x_i?

6. Alberto and Barbara are playing the following game. Initially, there are several piles of stones on the table. With Alberto playing first, a player in turn performs one of the following two moves:

   (a) take a stone from an arbitrary pile;

   (b) select a pile and divide it into two nonempty piles.

   The player who takes the last stone wins the game. Determine which player has a winning strategy in dependence of the initial state.