16-th Italian Mathematical Olympiad 2000

Cesenatico, May 5, 2000

1. A possitive integer is called *special* if all its decimal digits are equal and it can be represented as the sum of squares of three consecutive odd integers.

   (a) Find all 4-digit special numbers

   (b) Are there 2000-digit special numbers?

2. Let $ABCD$ be a convex quadrilateral. Denote $\angle DAB = \alpha$, $\angle ADB = \beta$, $\angle ACB = \gamma$, $\angle DBC = \delta$, $\angle DBA = \varepsilon$. Suppose $\alpha < 90^\circ$, $\beta + \gamma = 90^\circ$, $\delta + 2\varepsilon = 180^\circ$. Prove that
$$(DB + BC)^2 = AD^2 + AC^2.$$

3. A pyramid with the base $ABCD$ and the top V is inscribed in a sphere. Let $AD = 2BC$ and let the rays AB and DC intersect in point E. Compute the ratio of the volume of the pyramid $VAED$ to the volume of the pyramid $VABCD$.

4. Let $n > 1$ be a fixed integer. Alberto and Barbara play the following game:

   (i) Alberto chooses a positive integer;

   (ii) Barbara chooses an integer greater than 1 which is a multiple or submultiple of the number Alberto chose (including itself);

   (iii) Alberto increases or decreases the Barbara's number by 1.

   Steps (ii) and (iii) are alternatively repeated. Barbara wins if she succeeds to reach the number n in at most 50 moves. For which values of n can she win, no matter how Alberto plays?

5. A man disposes of sufficiently many metal bars of length 2 and wants to construct a grill of the shape of an $n \times n$ unit net. He is allowed to fold up two bars at an endpoint or to cut a bar into two equal pieces, but two bars may not overlap or intersect. What is the minimum number of pieces he must use?

6. Let $p(x)$ be a polynomial with integer coefficients such that $p(0) = 0$ and $0 \le p(1) \le 10^7$. Suppose that there exist positive integers a, b such that $p(a) = 1999$ and $p(b) = 2001$. Determine all possible values of $p(1)$. (*Note:* 1999 is a prime number.)

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