Joseph Gillis Mathematical Olympiad 1998

1. In space are given n segments A_iB_i and a point O not lying on any segment, such that the sum of the angles A_iOB_i is less than 180°. Prove that there exists a plane passing through O and not intersecting any of the segments.

2. Show that there is a multiple of 2^{1998} whose decimal representation consists only of the digits 1 and 2.

3. A configuration of several checkers at the centers of squares on a rectangular sheet of grid paper is called *boring* if some four checkers occupy the vertices of a rectangle with sides parallel to those of the sheet.

   (a) Prove that any configuration of more than $3mn/4$ checkers on an $m \times n$ grid is boring.

   (b) Prove that any configuration of 26 checkers on a 7×7 grid is boring.

4. A man has a seven-candle chandellier. The first evening he lighted one candle for one hour, the second evening he lighted two candles, also for one hour, and so on. After one hour the seventh evening, all seven candles simultaneously finished. How did the man choose the candles to light every evening?

5. (a) Find two real numebrs a, b such that $|ax + b - \sqrt{x}| \leq \frac{1}{24}$ for $1 \leq x \leq 4$.

   (b) Prove that the constant $1/24$ cannot be replaced by a smaller one.

6. Find all pairs (m, n) of integers with $m > n > 7$ for which there exists a polynomial $p(x)$ with integer coefficients such that $p(7) = 77$, $p(m) = 0$, and $p(n) = 85$.

7. A polygonal line of the length 1001 is given in a unit square. Prove that there exists a line parallel to one of the sides of the square that meets the polygonal line in at least 500 points.

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