Joseph Gillis Mathematical Olympiad 1997

February 17, 1997

1. Find all real solutions to the system of equations
$$x^2+y^2=6z,\quad y^2+z^2=6x,\quad z^2+x^2=6y.$$

2. We are given a balance with two bowls and a number of weights.
   (a) Give an example of four integer weights using which one can measure any weight of $1,2,\dots,40$ grams.
   (b) Are there four weights using which one can measure any weight of $1,2,\dots,50$ grams?

3. Let $n?$ denote the product of all primes smaller than n. Prove that $n? > n$ holds for any natural number $n>3$.

4. Let $f:[0,1]\to[0,1]$ be a continuous, strictly increasing function such that $f(0)=0$ and $f(1)=1$. Prove that
$$f\left(\frac{1}{10}\right)+f\left(\frac{2}{10}\right)+\cdots+f\left(\frac{9}{10}\right)+f^{-1}\left(\frac{1}{10}\right)+\cdots+f^{-1}\left(\frac{9}{10}\right)\le\frac{99}{10}.$$

5. The natural numbers $a_1,a_2,\dots,a_n$, $n\ge 12$, are smaller than $9n^2$ and pairwise coprime. Show that at least one of these numbers is prime.

6. In a certain country, every two cities are connected either by an airline route or by a railroad. Prove that one can always choose a type of transportation in such a way that each city can be reached from any other city with at most two transfers.

7. A square with side 10^6, with a corner square with side 10^{-3} cut off, is partitioned into 10 rectangles. Prove that at least one of these rectangles has the ratio of the greater side to the smaller one at least 9.

8. Two equal circles are internally tangent to a larger circle at A and B. Let M be a point on the larger circle, and let lines MA and MB intersect the corresponding smaller circles at A' and B'. Prove that $A'B'$ is parallel to AB.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com