Joseph Gillis Mathematical Olympiad 1996

1. Let a be a prime number and $n > 2$ an integer. Find all integer solutions of the equation $x^n + ay^n = a^2z^n$.

2. Find all polynomials $P(x)$ satisfying $P(x+1) - 2P(x) + P(x-1) = x$ for all x.

3. The angles of an acute triangle ABC at α, β, γ. Let AD be a height, CF a median, and BE the bisector of $\angle B$. Show that AD, CF and BE are concurrent if and only if $\cos\gamma \tan\beta = \sin\alpha$.

4. Eight guests arrive to a hotel with four rooms. Each guest dislikes at most three other guests and doesn't want to share a room with any of them (this feeling is mutual). Show that the guests can reside in the four rooms, with two persons in each room.

5. Suppose that the circumradius R and the inradius r of a triangle ABC satisfy $R = 2r$. Prove that the triangle is equilateral.

6. Let x, y, z be real numbers with $|x|, |y|, |z| > 2$. What is the smallest possible value of $|xyz + 2(x+y+z)|$?

7. Find all positive integers a, b, c such that
$$a^2 = 4(b+c) \quad \text{and} \quad a^3 - 2b^3 - 4c^3 = \frac{1}{2}abc.$$

8. Consider the function $f : \mathbb{N} \to \mathbb{N}$ given by
   (i) $f(1) = 1$,
   (ii) $f(2n) = f(n)$ for any $n \in \mathbb{N}$,
   (iii) $f(2n+1) = f(2n) + 1$ for any $n \in \mathbb{N}$.

   (a) Find the maximum value of $f(n)$ for $1 \le n \le 1995$;
   (b) Find all values of f on this interval.

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