Joseph Gillis Mathematical Olympiad 1995

1. Solve the system
$$\begin{aligned} x+\log\left(x+\sqrt{x^2+1}\right) &= y\\ y+\log\left(y+\sqrt{y^2+1}\right) &= z\\ z+\log\left(z+\sqrt{z^2+1}\right) &= x. \end{aligned}$$

2. Let H be a semicircle with diameter PQ. A circle O is internally tangent to H and touches diameter PQ at point C. Points A on H and B on PQ are such that AB is orthogonal to PQ and tangent to circle O. Prove that AC bisects $\angle PAB$.

3. If k and n are positive integers, prove the inequality
$$\frac{1}{kn}+\frac{1}{kn+1}+\cdots+\frac{1}{(k+1)n-1}\geq n\left(\sqrt[n]{\frac{k+1}{k}}-1\right).$$

4. Find all integers m and n satisfying $m^3-n^3-9mn=27$.

5. Let n be an odd positive integer and let $x_1,x_2,\dots,x_n$ be n distinct real numbers that satisfy $|x_i-x_j|\leq 1$ for $1\leq i<j\leq n$. Prove that
$$\sum_{i<j}|x_i-x_j|\leq\left[\frac{n}{2}\right]\left(\left[\frac{n}{2}\right]+1\right).$$

6. A 1995×1995 square board is given. A coloring of the cells of the board is called *good* if the cells can be rearranged so as to produce a colored square board that is symmetric with respect to the main diagonal. Find all values of k for which any k-coloring of the given board is good.

7. For certain n countries there is an airline connecting any two countries, but some of the airlines are closed. Show that if the number of the closed airlines does not exceed $n-3$, then one can make a round trip using the remaining airlines, starting from one of the countries, visiting every country exactly once and returning to the starting country.

8. A real number α is given. Find all functions $f:\mathbb{R}^+\to\mathbb{R}^+$ satisfying
$$\alpha x^2 f\left(\frac{1}{x}\right)+f(x)=\frac{x}{x+1}\quad\text{for all } x>0.$$