Grosman Memorial Mathematical Olympiad 1999

1. For any 16 positive integers $n, a_1, a_2, \ldots, a_{15}$ we define

$$T(n, a_1, a_2, \ldots, a_{15}) = (a_1^n + a_2^n + \cdots + a_{15}^n) a_1 a_2 \cdots a_{15}.$$

   Find the smallest n such that $T(n, a_1, \ldots, a_{15}$ is divisible by 15 for any choice of $a_1, \ldots, a_{15}$.

2. Find the smallest positive integer n for which $0 < \sqrt[4]{n} - [\sqrt[4]{n}] < 10^{-5}$.

3. For every triangle ABC, denote by $\mathscr{D}(ABC)$ the triangle whose vertices are the tangency points of the incircle of $\triangle ABC$ with the sides. Assume that $\triangle ABC$ is not equilateral.

   (a) Prove that $\mathscr{D}(ABC)$ is also not equilateral.

   (b) Find in the sequence $T_1 = \triangle ABC$, $T_{k+1} = \mathscr{D}(T_k)$ for $k \in \mathbb{N}$ a triangle whose largest angle α satisfies $0 < \alpha - 60^\circ < 0.0001^\circ$.

4. Consider a polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ with integer coefficients. Prove that if $f(x)$ has exactly one real root, then it can be factored into non-constant polynomials with rational coefficients.

5. An infinite sequence of distinct real numbers is given. Prove that it contains a subsequence of 1999 terms which is either monotonically increasing or monotonically decreasing.

6. Let A, B, C, D, E, F be points in space such that the quadrilaterals $ABDE$, $BCEF$, $CDFA$ are parallelograms. Prove that the six midpoints of the sides AB, BC, CD, DE, EF, FA are coplanar.

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