Grosman Mathematical Olympiad 1995

1. Positive integers $d_1, d_2, \dots, d_n$ are divisors of 1995. Prove that there exist d_i and d_j among them, such the denominator of the reduced fraction d_i/d_j is at least n.

2. Two players play a game on an infinite board that consists of unit squares. Player I chooses a square and marks it with O. Then player II chooses another square and marks it with X. They play until one of the players marks a whole row or a whole column of five consecutive squares, and this player wins the game. If no player can achieve this, the result of the game is a tie. Show that player II can prevent player I from winning.

3. Two thieves stole an open chain with $2k$ white beads and $2m$ black beads. They want to share the loot equally, by cutting the chain to pieces in such a way that each one gets k white beads and m black beads. What is the minimal number of cuts that is always sufficient?

4. Two given circles α and β intersect each other at two points. Find the locus of the centers of all circles that are orthogonal to both α and β.

5. For non-coplanar points are given in space. A plane π is called *equalizing* if all four points have the same distance from π. Find the number of equilizing planes.

6. (a) Prove that there is a unique function $f : \mathbb{Q} \to \mathbb{Q}$ satisfying:

   (i) $f(q) = 1 + f\left(\frac{q}{1-2q}\right)$ for $0 < q < \frac{1}{2}$;

   (ii) $f(q) = 1 + f(q-1)$ for $1 < q \le 2$;

   (iii) $f(q)f\left(\frac{1}{q}\right) = 1$ for all $q \in \mathbb{Q}^+$.

   (b) For this function f, find all $r \in \mathbb{Q}^+$ such that $f(r) = r$.

7. For a given positive integer n, let A_n be the set of all points (x,y) in the coordinate plane with $x, y \in \{0, 1, \dots, n\}$. A point (i, j) is called *internal* if $0 < i, j < n$. A real function f, defined on A_n, is called *good* if it has the following property: For every internal point x, the value of $f(x)$ is the arithmetic mean of its values on the four neighboring points (i.e. the points at the distance 1 from x). Prove that if f and g are good functions that coincide at the non-internal points of A_n, then $f \equiv g$.

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