Grosman Memorial Mathematical Olympiad 2001

1. Find all real solutions of the system

$$\begin{aligned} x_1 + x_2 + \cdots + x_{2000} &= 2000, \\ x_1^4 + x_2^4 + \cdots + x_{2000}^4 &= x_1^3 + x_2^3 + \cdots + x_{2000}^3. \end{aligned}$$

2. If $x_1, x_2, \ldots, x_{2001}$ are real numbers with $0 \le x_n \le 1$ for $n = 1, 2, \ldots, 2001$, find the maximum value of

$$\left(\frac{1}{2001}\sum_{n=1}^{2001} x_n^2\right) - \left(\frac{1}{2001}\sum_{n=1}^{2001} x_n\right)^2.$$

   Where is this maximum attained?

3. We are given 2001 lines in the plane, no two of which are parallel and no three of which are concurrent. These lines partition the plane into regions (not necessarily finite) bounded by segments of these lines. These segments are called *sides*, and the collection of the regions is called a *map*. Intersection points of the lines are called *vertices*. Two regions are *neighbors* if they share a side, and two vertices are neighbors if they lie on the same side. A *legal coloring* of the regions (resp. vertices) is a coloring in which each region (resp. vertex) receives one color, such that any two neighboring regions (vertices) have different colors.

   (a) What is the minimum number of colors required for a legal coloring of the regions?

   (b) What is the minimum number of colors required for a legal coloring of the vertices?

4. The lengths of the sides of triangle ABC are 4,5,6. For any point D on one of the sides, drop the perpendiculars DP, DQ onto the other two sides. What is the minimum value of PQ?

5. Triangle ABC in the plane Π is called *good* if it has the following property: For any point D in space outside the plane Π, it is possible to construct a triangle with sides of lengths CD, BD, AD. Find all good triangles.

6. (a) Find a pair of integers (x, y) such that $15x^2 + y^2 = 2^{2000}$.

   (b) Does there exist a pair of integers (x, y) such that $15x^2 + y^2 = 2^{2000}$ and x is odd?

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