16-th Iranian Mathematical Olympiad 1998/1999

Third Round

Time: 4 hours each day.

First Day

1. Suppose $A_1, A_2, \dots, A_k$ are distinct subsets of $X = \{1, 2, \dots, n\}$ such that for any positive integers $i_1 < i_2 < i_3 < i_4 \le k$ we have $|A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup A_{i_4}| \le n-2$. Prove that $k \le 2^{n-2}$.

2. Let a circle k through A and C intersects sides AB and BC at points D and E respectively. A circle with center S touches segments BD, BE and touches circle k at M. Prove that the bisector of $\angle ABC$ passes through the incenter of $\triangle ABC$.

3. Let $C_1, C_2, \dots, C_n$ be unit circles in the plane, no two of which are tangent, which form a connected set of points. If S is the set of all points that belong to at least two of the given circles, show that $|S| \ge n$.

Second Day

4. Let $x_1, \dots, x_n$ be real numbers from the interval $[-1, 1]$ such that $x_1 + \dots + x_n = 0$. Prove that there exists a permutation σ of the set $\{1, \dots, n\}$ such that for any integers p, q with $1 \le p \le q \le n$ we have
$$|x_{\sigma(p)} + x_{\sigma(p+1)} + \dots + x_{\sigma(q)}| \le 2 - \frac{1}{n}.$$
Prove that, if $2 - 1/n$ is replaced with $2 - 4/n$, the statement does not necessarily hold.

5. Let $ABCDEF$ be a convex hexagon such that $\angle B + \angle D + \angle F = 360^\circ$ and $\frac{AB}{BC} \cdot \frac{CD}{DE} \cdot \frac{EF}{FA} = 1$. Prove that
$$\frac{BC}{CA} \cdot \frac{AE}{EF} \cdot \frac{FD}{DB} = 1.$$

6. Let be given $r_1, \dots, r_n \in \mathbb{R}$. Show that there exists a subset I of $\{1, 2, \dots, n\}$ which meets each of the sets $\{i, i+1, i+2\}$ $(1 \le i \le n-2)$ in one or two elements such that
$$\left|\sum_{i \in I} r_i\right| \ge \frac{1}{6} \sum_{i=1}^{n} |r_i|.$$

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