14-th Iranian Mathematical Olympiad 1996/1997

Third Round

Time: 4 hours each day.

First Day

1. Let n be a positive integer. Prove that there exist polynomials $f(x)$ and $g(x)$ with integer coefficients such that
$$f(x)(x+1)^{2^n}+g(x)(x^{2^n}+1)=2.$$

2. Suppose that $f:\mathbb{R}\to\mathbb{R}$ has the following properties:
   (i) $f(x)\leq 1$ for all x;
   (ii) $f(x+\frac{13}{42})+f(x)=f(x+\frac{1}{6})+f(x+\frac{1}{7})$ for all x.

   Prove that f is periodic.

3. Let $w_1,w_2,\dots,w_k$ be distinct real numbers with a nonzero sum. Prove that there exist integers $n_1,n_2,\dots,n_k$ such that $\sum_{i=1}^k n_i w_i>0$, and for any non-identical permutation π of $\{1,2,\dots,k\}$, $\sum_{i=1}^k n_i w_{\pi(i)}<0$.

Second Day

4. Let P be a variable point on arc BC of the circumcircle of triangle ABC not containing A. Let I_1 and I_2 be the incenters of the triangles PAB and PAC respectively. Prove that:
   (a) The circumcircle of $\triangle PI_1I_2$ passes through a fixed point.
   (b) The circle with diameter I_1I_2 passes through a fixed point.
   (c) The midpoint of I_1I_2 lies on a fixed circle.

5. Suppose that $f:\mathbb{R}^+\to\mathbb{R}^+$ is a function such that for all $x,y>0$
$$f(x+y)+f(f(x)+f(y))=f(\,f(x+f(y))+f(y+f(x))\,).$$
   Prove that $f(x)=f^{-1}(x)$.

6. A building consists of finitely many rooms which have been separated by walls. There are some doors on some of these walls which can be used to go around the building. Assume it is possible to reach any room from any other room. Two fixed rooms are marked by S and E. A person starts walking from S and wants to reach E.

   A *program* $P=(P_i)_{i\in I}$ is an R,L-sequence. The person uses it as follows: After passing through the n-th door, he chooses the door just to the right or left from

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

the door just passed, meaning that P_n is R or L, and gets through it. In a room with one door, any symbol means selecting the door he has just passed. The person stops as soon as he reaches E.

Prove that there is a (possibly infinite) program P with the property that, no matter how the structure of the building is, the person can reach E by following it.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com