23-rd Iranian Mathematical Olympiad 2005/06

Third Round

Time: 4.5 hours each day.

First Day

1. Suppose that the circumradius of a triangle ABC is equal to its exradius across A. The excircle across A touches BC, CA, AB at M, N, L, respectively. Show that the orthocenter of triangle MNL coincides with the circumcenter O of $\triangle ABC$.

2. Let $x_1, x_2, \dots, x_n$ be real numbers. Prove that
$$\sum_{i,j=1}^{n} |x_i + x_j| \geq n \sum_{i=1}^{n} |x_i|.$$

3. Each edge of a complete oriented graph G is colored red or blue. Show that there is a vertex v of G from which there is a monochromatic (directed) path to any other vertex.

Second Day

4. On the plane are given n points, no three on a line. A set of these points is called *polite* if they form a convex polygon with no points inside. Let c_k denote the number of polite sets with k points. Show that the sum
$$\sum_{i=3}^{n} (-1)^i c_i$$
only depends on n and not on the configuration of the points.

5. Given an integer $n > 1$, find all n-tuples of distinct, pairwise coprime natural numbers $a_1, \dots, a_n$ such that $a_1 + \dots + a_n$ divides $a_1^i + \dots + a_n^i$ for $1 \leq i \leq n$.

6. Show that an arbitrary simple (not necessarily convex) polygon has a diagonal which is completely inside the polygon and divides the perimeter into two parts which each have at least one third of the vertices of the polygon.

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