17-th Iranian Mathematical Olympiad 1999/2000

Third Round

Time: 4 hours each day.

First Day

1. In a tennis tournament where n players $A_1, A_2, \dots, A_n$ take part, any two players play at most one match, and $k \leq \frac{n(n-1)}{2}$ matches are played. The winner of a match gets 1 point while the loser gets 0. Prove that a sequence $d_1, d_2, \dots, d_n$ of nonnegative integers can be the sequence of scores of the players (d_i being the score of A_i) if and only if

   (i) $d_1 + d_2 + \cdots + d_n = k$, and

   (ii) for any $X \subset \{A_1, \dots, A_n\}$, the number of matches between the players in X is at most $\sum_{A_j \in X} d_j$.

2. Isosceles triangles $A_3A_1O_2$ and $A_1A_2O_3$ are constructed on the sides of a triangle $A_1A_2A_3$ as the bases, outside the triangle. Let O_1 be a point outside $\triangle A_1A_2A_3$ such that
$$\angle O_1A_3A_2 = \frac{1}{2}\angle A_1O_3A_2 \quad \text{and} \quad \angle O_1A_2A_3 = \frac{1}{2}\angle A_1O_2A_3.$$
Prove that $A_1O_1 \perp O_2O_3$, and if T is the projection of O_1 onto A_2A_3, then $A_1O_1/O_2O_3 = 2O_1T/A_2A_3$.

3. A circle Γ with radius R and center ω, and a line d are drawn on a plane, such that the distance of ω from d is greater than R. Two points M and N vary on d so that the circle with diameter MN is tangent to Γ. Prove that there is a point P in the plane from which all the segments MN are visible at a constant angle.

Second Day

4. Let n be a positive integer. Suppose S is a set of ordered n-tuples of nonnegative integers such that, whenever $(a_1, \dots, a_n) \in S$ and b_i are nonnegative integers with $b_i \leq a_i$, the n-tuple $(b_1, \dots, b_n)$ is also in S. If h_m is the number of elements of S with the sum of components equal to m, prove that h_m is a polynomial in m for all sufficiently large m.

5. Suppose that a, b, c are real numbers such that for all positive numbers $x_1, x_2, \dots, x_n$ we have
$$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^a \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^b \left(\frac{1}{n}\sum_{i=1}^{n} x_i^3\right)^c \geq 1.$$
Prove that vector (a, b, c) is a nonnegative linear combination of vectors $(-2, 1, 0)$ and $(-1, 2, -1)$.

6. Prove that for every natural number n there exists a polynomial $p(x)$ with integer coefficients such that $p(1), p(2), \dots, p(n)$ are distinct powers of 2.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imo.org.yu