16-th Iranian Mathematical Olympiad 1998/1999

Second Round

Time: 4 hours each day.

First Day

1. Define the sequence (x_n) by $x_0 = 1$ and for all $n \in \mathbb{N}$,

$$x_n = \begin{cases} x_{n-1} + (3^r - 1)/2 & \text{if} \quad n = 3^{r-1}(3k+1); \\ x_{n-1} - (3^r + 1)/2 & \text{if} \quad n = 3^{r-1}(3k+2). \end{cases},$$

where $k \in \mathbb{N}_0$, $r \in \mathbb{N}$. Prove that every integer occurs in this sequence exactly once.

2. Let $n(r)$ be the maximum possible number of points with integer coordinates on a circle with radius r in Cartesian plane. Prove that $n(r) < 6\sqrt[3]{\pi r^2}$.

3. Let $ABCDEF$ be a convex hexagon such that $AB = BC$, $CD = DE$ and $EF = FA$. Prove that

$$\frac{AB}{AD} + \frac{CD}{CF} + \frac{EF}{EB} \geq \frac{3}{2}.$$

Second Day

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x, y,

$$f(f(x) + y) = f(x^2 - y) + 4f(x)y.$$

5. In a triangle ABC, the bisector of angle BAC intersects BC at D. The circle Γ through A which is tangent to BC at D meets AC again at M. Line BM meets Γ again at P. Prove that line AP is a median of $\triangle ABD$.

6. Let ABC be a given triangle. Consider any painting of points of the plane in red and green. Show that there exist either two red points on the distance 1, or three green points forming a triangle congruent to $\triangle ABC$.

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