14-th Iranian Mathematical Olympiad 1996/1997

Second Round

Time: 4 hours each day.

First Day

1. Suppose that S is a finite set of real numbers with the property that any two distinct elements of S form an arithmetic progression with another element in S. Give an example of such a set with 5 elements and show that no such set exists with more than 5 elements.

2. Suppose that 10 points are given in the plane, such that among any five of them there are four lying on a circle. Find the minimum number of these points which must lie on a circle.

3. Let Γ be a semicircle with center O and diameter AB. Let M be a point on the extension of AB such that $MA > MB$. A line through M meets Γ at C and D such that $MC > MD$. The circumcircles of the triangles AOC and BOD meet at O and K. Prove that $OK \perp MK$.

Second Day

4. Determine all functions $f : \mathbb{N}_0 \to \mathbb{N}_0 \setminus \{1\}$ such that for all $n > 0$
$$f(n+1) + f(n+3) = f(n+5)f(n+7) - 1375.$$

5. In an acute-angled triangle ABC, O, H, and P are the circumcenter, orthocenter and the foot of the altitude from C, respectively. The line perpendicular to OP at P intersects the line AC at Q. Prove that $\angle PHQ = \angle BAC$.

6. Let A be a symmetric $\{0,1\}$-matrix with all the diagonal entries equal to 1. Show that there exist indices $0 \le i_1 < i_2 < \cdots < i_k \le n$ such that
$$A_{i_1} + A_{i_2} + \cdots + A_{i_k} = (1, 1, \ldots, 1) \pmod 2,$$
where A_i denotes the i-th column of A.

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