24-th Iranian Mathematical Olympiad 2006/2007

Second Round

Time: 4 hours each day.

First Day

1. A regular polyhedron is a polyhedron which is convex and all of its faces are regular polygons. We call a regular polyhedron a *TLP*, if and only if none of its faces is a triangle.

   (a) Prove that each *TLP* can be inscribed in a sphere.

   (b) Prove that the faces of each *TLP* are polygons of at most 3 different kinds. (i.e. there is a set $\{m,n,p\}$ such that each face of the TLP is a regular n-gon or m-gon or p-gon.)

   (c) Prove that there is only on *TLP* with only pentagonal and hexagonal faces. (Soccer ball)

   (d) For $n > 3$, a prism which has 2 regular n-gons and n squares as its faces, is a *TLP*. Prove that except for these *TLP*s there are finitely many other *TLP*s.

2. A fluid is flowing in an infinite, line-shaped pipe. For each molecule of the fluid, if it is at the point with coordinate x, after t seconds it will be at the point with coordinate $P(t,x)$. Prove that if $P(t,x)$ is a polynomial of t, x, then all molecules are moving with a unique and constant speed.

3. Suppose that C is a convex subset of $\mathbb{R}^3$ with positive volume. Suppose that $C_1,\dots,C_n$ are n translated (not rotated) copies of C such that $C_i \cap C \neq \emptyset$, but C_i and C_j intersect at most on the boundary for $i \neq j$. Prove that $n \leq 27$ and also prove that 27 is the best bound.

   (a) Prove the above for symmetric C.

   (b) Prove the above for arbitrary C.

Second Day

4. We have a finite number of disjoint shapes in a plane. A *Convex Partitioning* is a partitioning of the plane into convex parts, such that each part contains exactly one of the shapes. The parts can intersect at most on the boundary and they must cover the plane. For which of the following sets of shapes, a *Convex Partitioning* exists?

   (a) A finite number of distinct points;

   (b) A finite number of disjoint line segments;

   (c) A finite number of disjoint circular disks.

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5. For $A \subseteq \mathbb{Z}$ and $a, b \in \mathbb{Z}$ let $aA+b$ denote the set $\{ax+b : x \in \mathbb{A}\}$. If $a \neq 0$ then we say that $aA+b$ is similar to A. The *Cantor set* C is the set of non-negative integers which have no digit 1 in their base 3 representation. One can see that $C = (3C) \cup (3C+2)$ and that this union is disjoint. Another example is $C = (3C) \cup (9C+6) \cup (3C+2)$ and this union is disjoint as well. A representation of C is a partitioning of C into a finite number (bigger than 1) of sets similar to C, i.e.

$$C = \bigcup_{i=1}^{n} C_i$$

where $C_i = a_i C + b_i$. We call a representation of C a primitive representation, if and only if the union of some of the C_is is not a set similar but different than C. Consider the primitive representation of the Cantor set. Prove that

(a) $a_i > 1$.

(b) Each a_i is a power of 3.

(c) $a_i > b_i$.

(d) The only primitive representation of C is the following one:

$$C = (3C) \cup (3C+2).$$

6. Let $P(x)$ and $Q(x)$ be the polynomials with integer coefficients. If $P(x)$ is monic, prove that there exists a monic polynomial $R(x) \in \mathbb{Z}[x]$ such that

$$P(x) \mid Q(R(x)).$$

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