21-st Iranian Mathematical Olympiad 2003/04

Second Round

Time: 4.5 hours each day.

First Day

1. Let P and Q be points on the sides BC and DC respectively of a convex quadrilateral $ABCD$ such that $\angle BAP = \angle DAQ$. Prove that the areas of triangles ABP and ADQ are equal if and only if the line through the orthocenters of these triangles is perpendicular to AC.

2. Let $f_1, f_2, \dots, f_n$ be polynomials with integer coefficients. Show that there exists a reducible (in $\mathbb{Z}[x]$) polynomial $g(x)$ with integer coefficients such that $f_i(x) + g(x)$ is irreducible for $i = 1, \dots, n$

3. Let X be a set of n elements and $0 \le k \le n$ be an integer. We denote by $a_{n,k}$ ($b_{n,k}$) the maximum possible number of permutations of X every two of which match in at least (resp. at most) k positions.

   (a) Show that $a_{n,k} b_{n,k-1} \le n!$.

   (b) For a prime number p find the exact value of $a_{p,2}$.

Second Day

4. Does there exist an infinite set $S \in \mathbb{N}$ such that for any $a, b \in S$, $a^2 - ab + b^2$ divides $(ab)^2$.

5. A light-point is placed in space. Is it possible to block the light with a finite number of disjoint spheres of the same size?

6. The sides of a given n-gon $\mathcal{P}$ are numbered by 1 through n. For a sequence $S = (s_1, s_2, s_3, \dots)$ with $s_i \in \{1, \dots, n\}$, polygon $\mathcal{P}$ moves around the plane as follows: In the i-th step, it reflects in its side numbered by s_i.

   (a) Show that there is an infinite sequence S such that by moving $\mathcal{P}$ according to S we can cover every point in the plane at least once.

   (b) Prove that such a sequence cannot be periodic.

   (c) If $\mathcal{P}$ is a regular polygon of circumradius 1 and D an arbitrary circle of radius 1.0001 in the plane, does there necessarily exist a finite sequence S that will place $\mathcal{P}$ inside the circle D?

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