19-th Iranian Mathematical Olympiad 2001/02

Second Round

Time: 4.5 hours each day.

First Day

1. The sequence (a_n) is defined by $a_0 = 2$, $a_1 = 1$, and $a_{n+1} = a_n + a_{n-1}$ for $n \geq 1$. Show that if P is a prime factor of $a_{2k} - 2$ for some $k \in \mathbb{N}$, then P is a factor of $a_{2k+1} - 1$.

2. Let A be a point outside the circle Ω. The tangents from A to Ω touch Ω at B and C. A tangent L to Ω intersects AB at P and AC at Q. The line through P parallel to AC meets BC at R. Prove that as L varies, QR passes through a fixed point.

3. An ant moves on a straight path on the surface of a cube. If the ant reaches an edge, it goes on in such a way that if the cube were opened to make the adjacent faces coplanar, the path would become a straight line. If the ant reaches a vertex, it returns on the same path.

   (a) Show that for every starting point of the ant, there are infinitely many directions for the ant to move in a periodic path.

   (b) Show that if the ant starts on a fixed face, the periodicity of the path depends only on the direction (not the starting point).

Second Day

4. Find the smallest positive integer n for which the following condition holds: For every finite set of points in the plane, if, for every n points in this set, there exist two lines covering all n points, then there exist two lines covering all points in the set.

5. In triangle ABC, the incircle with center I touches AB at X and AC at Y. The line XI meets the incircle again at M. Let X' be the point of intersection of AB and CM. Point L is on the segment $X'C$ such that $X'L = CM$. Prove that A, L, and Y are collinear if and only if $AB = AC$.

6. Positive numbers a, b, c satisfy $a^2+b^2+c^2+abc = 4$. Prove that $a+b+c \leq 3$.

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