18-th Iranian Mathematical Olympiad 2000/01

Second Round

Time: 4 hours each day.

First Day

1. Prove that every real number α with $1 < \alpha < 2$ has a unique representation as an infinite product

$$\alpha = \left(1 + \frac{1}{n_1}\right)\left(1 + \frac{1}{n_2}\right)\cdots,$$

where n_i are natural numbers with $n_i^2 \le n_{i+1}$.

2. Denote $\Delta^3 = \{x \in \mathbb{R}^3 \mid \|x\| \le 1\}$ and $\Delta^2 = \{x \in \mathbb{R}^2 \mid \|x\| \le R\}$. Does there exist a function $f : \Delta^3 \to \Delta^2$ such that

$$\|f(A) - f(B)\| \ge \|A - B\| \quad \text{for all } A, B \in \Delta^3?$$

3. Let $x_1, x_2, \dots$ be a random sequence of 0 and 1. Which of the sequences 110 and 010 is more probable to occur first?

Second Day

4. Let x, y, z be positive integers with $xy = z^2 + 1$. Prove that there exist integers a, b, c, d such that

$$x = a^2 + b^2, \quad y = c^2 + d^2, \quad z = ac + bd.$$

5. Let B and D be points on segments AC and AE respectively, and let CD and BE intersect at F. If $AB + BF = AD + DF$, prove that $AC + CF = AE + EF$.

6. Suppose that (a_n) is a sequence of positive integers such that for all $m, n \in \mathbb{N}$ we have $\gcd(a_m, a_n) = a_d$, where $d = \gcd(m, n)$. Prove that there exists a sequence (b_n) of positive integers such that

$$a_n = \prod_{d|n} a_d.$$

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