16-th Iranian Mathematical Olympiad 1998/1999

First Round

Time: 3 hours each day.

First Day

1. Let $a_1 < a_2 < \cdots < a_n$ be real numbers. Prove that
$$a_1a_2^4 + a_2a_3^4 + \cdots + a_na_1^4 \geq a_1^4a_2 + a_2^4a_3 + \cdots + a_n^4a_1.$$

2. Given a natural number n, an n-tuple $(a_1, \dots, a_n)$ is said to be *good* if $a_1 + \cdots + a_n = 2n$ and for no subset S of $\{1, 2, \dots, n\}$, $\sum_{i \in S} a_i = n$. Find all good n-tuples.

3. Let I be the incenter of a triangle ABC and D the point of intersection of line AI with the circumcircle of $\triangle ABC$. Points D and E are respectively the feet of perpendiculars from I to BD and CD. Given that $IE + IF = AD/2$, compute $\angle BAC$.

Second Day

4. In a triangle ABC we have $BC > CA > AB$. Points D and E are given on side BC and ray AB respectively so that $BD = BE = AC$. The circumcircle of triangle BED meets AC at P. Line BP meets the circumcircle of ABC at Q. Show that $AQ + CQ = BP$.

5. Let $d_1 < d_2 < d_3 < d_4$ be the four smallest divisors of a natural number n. Find all n such that $n = d_1^2 + d_2^2 + d_3^2 + d_4^2$.

6. For any two n-sequences of 0 and 1 $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_n)$, we define their distance by $d(A,B) = |\{i \mid a_i \neq b_i\}|$. Suppose A, B, C are such sequences that satisfy $d(A,B) = d(B,C) = d(C,A) = \delta$.

   (a) Prove that δ is even.

   (b) Prove that there is a sequence D such that $d(D,A) = d(D,B) = d(D,C) = \delta/2$.

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