24-th Iranian Mathematical Olympiad 2006/2007

First Round

Time: 4 hours each day.

First Day

1. For each $m,n>2$ prove that there exists a sequence $a_0,\dots,a_k$ such that $a_0=m$, $a_k=n$ and
$$a_i+a_{i+1}|a_ia_{i+1}+1,\ \ i=0,1,\dots,k-1.$$

2. Let $I_1,\dots,I_n$ be n closed intervals of $\mathbb{R}$ such that among any k of them there are 2 with nonempty intersection. Prove that one can choose $k-1$ points in $\mathbb{R}$ such that any of the intervals contain at least one of the chosen points.

3. Let A,B,C,D be four points in the alphabetical order on a circle Ω. Prove that there are four points M_1,M_2,M_3,M_4 on the circle which form a quadrilateral with perpendicular diagonals, such that for each $i\in\{1,2,3,4\}$
$$\frac{AM_i}{BM_i}=\frac{DM_i}{CM_i}.$$

Second Day

4. Fomd all polynomials $p(x,y)\in\mathbb{R}[x,y]$ such that
$$\forall x,y\in\mathbb{R}:\ p(x+y,x-y)=2p(x,y).$$

5. Let C_1 and C_2 be two circles such that the center of C_1 is located on C_2. If M and N are the intersections of the circles, AB an arbitrary diameter of C_1, A_1 and B_1 the intersections of AM and BM with C_2 respectively, prove that A_1B_1 is equal to the radius of C_1.

6. We have a stack of n books piled on each other, and labeled by $1,2,\dots,n$. In each round we make n moves in the following manner: In the i-th move of each turn, we turn over the i books at the top, as a single book. After each round we start a new round similar to the previous one. Show that after some moves, we will reach the initial arrangement.

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