23-rd Iranian Mathematical Olympiad 2005/06

First Round

Time: 4 hours each day.

First Day

1. Suppose that n is a positive integer and p a prime number such that $n \mid p-1$ and $p \mid n^3-1$. Show that $4p-3$ is a perfect square.
2. Let D be a variable point on side BC of a triangle ABC with $\angle A = 60^\circ$. Let O_1 and O_2 be the circumcenters of triangles ABD and ACD respectively, and let N be the circumcenter of triangle DO_1O_2. The lines BO_1 and CO_2 intersect at M. Prove that the line MN passes through a fixed point.
3. Given 10^6 points in space, show that the set of their pairwise distances has at least 79 elements.

Second Day

4. In some of the $2n$ cells of a $2 \times n$ table there are (one or more) coins. In each step we choose a cell with at least two coins, remove two coins and put one either on the upper cell, or on the cell to the right. If we start with at least 2^n coins on the table, prove that we can play so that we bring at least one coin to the upper-right cell.
5. A chord XY of a circle is perpendicular to its diameter BC. Points P and M are taken on XY and CY so that $CY \parallel PB$ and $CX \parallel MP$. The lines CX and PB intersect at K. Prove that PB is perpendicular to MK.
6. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(x+y)f(f(x)y) = x^2 f(f(x)+f(y)) \quad \text{for all } x, y > 0.$$

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