21-st Iranian Mathematical Olympiad 2003/04

First Round

Time: 4.5 hours each day.

First Day

1. A natural number n is called *3-partite* if the set of its divisors can be partitioned into three subsets with the same sum of elements.
   (a) Find at least one 3-partite number.
   (b) Show that there exist infinitely many 3-parite numbers.
2. On the plane are given $n \geq 3$ points which are not all collinear. For an arbitrary point A in the plane we define $S(A)$ to be the sum of the distances from A to the n points. A point A is called *good* if for any other point B in the plane we have $S(B) \geq S(A)$. Prove that there is at most one good point.
3. There are n teams in a volleyball tournemant, where any two teams played one match. Suppose that for any two teams A and B there are exactly t teams that were defeated by both A and B. Prove that $n = 4t + 3$.

Second Day

4. If x, y, z are real numbers with $xyz = -1$, prove that
$$x^4 + y^4 + z^4 + 3(x + y + z) \geq \frac{x^2}{y} + \frac{x^2}{z} + \frac{y^2}{x} + \frac{y^2}{z} + \frac{z^2}{x} + \frac{z^2}{y} .$$
5. In a triangle ABC the smallest angle is at A. Let D be a point on the shorter arc BC of the circumcircle of $\triangle ABC$. The perpendicular bisectors of AB and AC intersect AD at M and N, respectively. Lines BM and CN meet at T. If R denotes the circumradius of $\triangle ABC$, prove that $BT + CT \leq 2R$.
6. A robot is moving on the edges of an infinite chessboard whose cells are unit squares. There are two counters A and B which are initially both set to zero. In each step the robot moves one unit to the north, south, west, or east, and depending on the type of the step A is replaced by $A + 1$, A by $A - 1$, B by $B - A$, or B by $B + A$, respectively. Assume that the robot starts its journey at a vertex V of the chessboard and that it returns to V after some time without intersecting its trajectory. Prove that the final value of $|B|$ will be equal to the area of the region enclosed by the trajectory.

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