19-th Iranian Mathematical Olympiad 2001/02

First Round

Time: 4.5 hours each day.

First Day

1. Find all permutations $(a_1, \dots, a_n)$ of $1, 2, \dots, n$ which have the property that $i+1$ divides $2(a_1 + \cdots + a_i)$ for every $i = 1, \dots, n$.

2. A rectangle is partitioned into finitely many small rectangles. We call a point a *cross point* if it belongs to four different small rectangles. We call a segment on the obtained diagram *maximal* if there is no other segment containing it. Show that the number of maximal segments plus the number of cross points is 3 more than the number of small rectangles.

3. In a convex quadrilateral $ABCD$ with $\angle ABC = \angle ADC = 135^\circ$, points M and N are taken on the rays AB and AD respectively such that $\angle MCD = \angle NCB = 90^\circ$. The circumcircles of triangles AMN and ABD intersect at A and K. Prove that $AK \perp KC$.

Second Day

4. Let A and B be two fixed points in the plane. Consider all possible convex quadrilaterals $ABCD$ with $AB = BC$, $AD = DC$, and $\angle ADC = 90^\circ$. Prove that there is a fixed point P such that, for every such quadrilateral $ABCD$ on the same side of AB, the line DC passes through P.

5. Let δ be a symbol such that $\delta \neq 0$ and $\delta^2 = 0$. Define $\mathbb{R}[\delta] = \{a + b\delta \mid a, b \in \mathbb{R}\}$, where $a + b\delta = c + d\delta$ if and only if $a = c$ and $b = d$, and define

$$\begin{aligned}(a+b\delta)+(c+d\delta) &= (a+c)+(b+d)\delta,\\ (a+b\delta)\cdot(c+d\delta) &= ac+(ad+bc)\delta.\end{aligned}$$

   Let $P(x)$ be a polynomlal with real coefficients. Show that $P(x)$ has a multiple real root if and only if $P(x)$ has a non-real root in $\mathbb{R}[\delta]$.

6. Let G be a simple graph with 100 edges on 20 vertices. Suppose that we can choose a pair of disjoint edges in 4050 ways. Prove that G is regular.

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