17-th Iranian Mathematical Olympiad 1999/2000

First Round

Time: 4 hours each day.

First Day

1. Does there exist a natural number N which is a power of 2, such that one can permute its decimal digits to obtain a different power of 2?

2. Three non-coplanar circles are given in the Euclidean space so that they are pairwise tangent. Prove that there exists a sphere that passes through all the three circles.

3. In a deck of $n > 1$ cards, some digits from 1 to 8 are written on each card. A digit may occur more than once, but at most once on a certain card. On each card at least one digit is written, and no two cards are denoted by the same set of digits. Suppose that for every $k = 1, 2, \ldots, 7$ digits, the number of cards that contain at least one of them is even. Find n.

Second Day

4. A sequence of natural numbers $c_1, c_2, \ldots$ is called *perfect* if every natural number m with $1 \leq m \leq c_1 + \cdots + c_n$ can be represented as

$$m = \frac{c_1}{a_1} + \frac{c_2}{a_2} + \cdots + \frac{c_n}{a_n}, \quad a_i \in \mathbb{N}.$$

Given n, find the maximum possible value of c_n in a perfect sequence (c_i).

5. Circles C_1 and C_2 with centers at O_1 and O_2 respectively meet at points A and B. The radii O_1B and O_2B meet C_1 and C_2 at F and E. The line through B parallel to EF intersects C_1 again at M and C_2 again at N. Prove that $MN = AE + AF$.

6. Two triangles ABC and $A'B'C'$ are positioned in the space such that the length of every side of $\triangle ABC$ is not less than a, and the length of every side of $\triangle A'B'C'$ is not less than a'. Prove that one can select a vertex of $\triangle ABC$ and a vertex of $\triangle A'B'C'$ so that the distance between the two selected vertices is not less than $\sqrt{\dfrac{a^2 + a'^2}{3}}$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imo.org.yu