12-th Irish Mathematical Olympiad 1999

May 8, 1999

Time: 3 hours each part.

Part 1

1. Find all real numbers x which satisfy

$$\frac{x^2}{\left(x+1-\sqrt{x+1}\right)^2} < \frac{x^2+3x+18}{(x+1)^2}.$$

2. Show that there is a positive number in the Fibonacci sequence which is divisible by 1000.

3. If AD is the altitude, BE the angle bisector, and CF the median of a triangle ABC, prove that AD, BE, and CF are concurrent if and only if

$$a^2(a-c) = (b^2-c^2)(a+c),$$

where a, b, c are the lengths of the sides BC, CA, AB, respectively.

4. A 100×100 square floor consisting of 10000 squares is to be tiled by rectangular 1×3 tiles, fitting exactly over three squares of the floor.

   (a) If a 2×2 square is removed from the center of the floor, prove that the rest of the floor can be tiled with the available tiles.

   (b) If, instead, a 2×2 square is removed from the corner, prove that such a tiling is not possible.

5. The sequence u_n, $n = 0, 1, 2, \dots$ is defined by $u_0 = 0$, $u_1 = 1$ and for each $n \geq 1$, u_{n+1} is the smallest positive integer greater than u_n such that $\{u_0, u_1, \dots, u_{n+1}\}$ contains no three elements in arithmetic progression. Find u_{100}.

Part 2

6. Solve the system of equations

$$y^2 = (x+8)(x^2+2),$$
$$y^2 - (8+4x)y + (16+16x-5x^2) = 0.$$

7. A function $f : \mathbb{N} \to \mathbb{N}$ satisfies

   (i) $f(ab) = f(a)f(b)$ whenever a and b are coprime;

   (ii) $f(p+q) = f(p) + f(q)$ for all prime numbers p and q.

   Prove that $f(2) = 2$, $f(3) = 3$ and $f(1999) = 1999$.

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8. The sum of positive real numbers a,b,c,d is 1. Prove that

$$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+d}+\frac{d^2}{d+a}\geq\frac{1}{2},$$

with equality if and only if $a=b=c=d=\frac{1}{4}$.

9. Find all positive integers m with the property that the fourth power of the number of (positive) divisors of m equals m.

10. A convex hexagon $ABCDEF$ satisfies $AB=BC$, $CD=DE$, $EF=FA$ and

$$\angle ABC+\angle CDE+\angle EFA=360^\circ.$$

Prove that the perpendiculars from A,C and E to FB,BD and DF respectively are concurrent.

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