11-th Irish Mathematical Olympiad 1998

May 9, 1998

Time: 3 hours each part.

Part 1

1. Prove that if $x \neq 0$ is a real number, then $x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \geq 0$.

2. The distances from a point P inside an equilateral triangle to the vertices of the triangle are $3, 4$, and 5. Find the area of the triangle.

3. Show that no integer of the form $\overline{xyxy}$ in base 10 can be a perfect cube. Find the smallest base $b > 1$ for which there is a perfect cube of the form $\overline{xyxy}$ in base b.

4. Show that a disk of radius 2 can be covered by seven (possibly overlapping) disks of radius 1.

5. If x is a real number such that $x^2 - x$ and $x^n - x$ are integers for some $n \geq 3$, prove that x is an integer.

Part 2

6. Find all positive integers n having exactly 16 divisors $1 = d_1 < d_2 < \cdots < d_{16} = n$ such that $d_6 = 18$ and $d_9 - d_8 = 17$.

7. Prove that if a, b, c are positive real numbers, then
$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

8. (a) Prove that $\mathbb{N}$ can be partitioned into three (mutually disjoint) sets such that, if $m, n \in \mathbb{N}$ and $|m-n|$ is 2 or 5, then m and n are in different sets.

   (b) Prove that $\mathbb{N}$ can be partitioned into four sets such that, if $m, n \in \mathbb{N}$ and $|m-n|$ is 2, 3 or 5, then m and n are in different sets. Show, however, that $\mathbb{N}$ cannot be partitioned into three sets with this property.

9. A sequence (x_n) is given as follows: x_0, x_1 are arbitrary positive real numbers, and $x_{n+2} = \frac{1 + x_{n+1}}{x_n}$ for $n \geq 0$. Find x_{1998}.

10. A triangle ABC has integer sides, $\angle A = 2\angle B$ and $\angle C > 90^\circ$. Find the minimum possible perimeter of this triangle.