10-th Irish Mathematical Olympiad 1997

May 10, 1997

Time: 3 hours each part.

Part 1

1. Find all pairs of integers (x,y) satisfying $1+1996x+1998y=xy$.
2. For a point M inside an equilateral triangle ABC, let D,E,F be the feet of the perpendiculars from M onto BC,CA,AB, respectively. Find the locus of all such points M for which $\angle FDE$ is a right angle.
3. Find all polynomials $p(x)$ satisfying the equation
$$(x-16)p(2x)=16(x-1)p(x) \quad \text{for all } x.$$
4. Let a,b,c be nonnegative real numbers. Suppose that $a+b+c\geq abc$. Prove that
$$a^2+b^2+c^2\geq abc.$$
5. Let S be the set of odd integers greater than 1. For each $x\in S$, denote by $\delta(x)$ the unique integer satisfying the inequality $2^{\delta(x)}<x<2^{\delta(x)+1}$. For $a,b\in S$, define
$$a*b=2^{\delta(a)-1}(b-3)+a.$$
Prove that if $a,b,c\in S$, then
   (a) $a*b\in S$ and
   (b) $(a*b)*c=a*(b*c)$.

Part 2

6. Given a positive integer n, denote by $\sigma(n)$ the sum of all positive divisors of n. We say that n is *abundant* if $\sigma(n)>2n$. (For example, 12 is abuntant since $\sigma(12)=28>2\cdot 12$.) Let a,b be positive integers and suppose that a is abundant. Prove that ab is abundant.
7. A circle Γ is inscribed in a quadrilateral $ABCD$. If
$$\angle A=\angle B=120^\circ,\ \angle D=90^\circ \quad \text{and} \quad BC=1,$$
find, with proof, the length of AD.
8. Let A be a subset of $\{0,1,2,\dots,1997\}$ containing more than 1000 elements. Prove that either A contains a power of 2 (that is, a number of the form 2^k with $k=0,1,2,\dots$) or there exist two distinct elements $a,b\in A$ such that $a+b$ is a power of 2.

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9. Let S be the set of natural numbers n satisfying the followiing conditions:

   (i) n has 1000 digits,

   (ii) all the figits of n are odd, and

   (iii) any two adjacent digits of n differ by 2.

   Determine the number of elements of S,

10. Let p be an odd prime number and n a natural number. Then n is called *p-partitionable* if $T = \{1, 2, \dots, n\}$ can be partitioned into (disjoint) subsets $T_1, T_2, \dots, T_p$ with equal sums of elements. For example, 6 is 3-partitionable since we can take $T_1 = \{1, 6\}$, $T_2 = \{2, 5\}$ and $T_3 = \{3, 4\}$.

    (a) Suppose that n is p-partitionable. Prove that p divides n or $n+1$.

    (b) Suppose that n is divisible by $2p$. Prove that n is p-partitionable.