7-th Irish Mathematical Olympiad 1994

May 7, 1994

Time: 3 hours each part.

Part 1

1. Let x, y be positive integers with $y > 3$ and $x^2 + y^4 = 2((x-6)^2 + (y+1)^2)$. Prove that $x^2 + y^4 = 1994$.

2. Let A, B, C be collinear points on the plane with B between A and C. Equilateral triangles ABD, BCE, CAF are constructed with D, E on one side of the line AC and F on the other side. Prove that the centroids of the triangles are the vertices of an equilateral triangle, and that the centroid of this triangle lies on the line AC.

3. Find all real polynomials $f(x)$ satisfying $f(x^2) = f(x)f(x-1)$ for all x.

4. Consider all $m \times n$ matrices whose all entries are 0 or 1. Find the number of such matrices for which the number of 1-s in each row and in each column is even.

5. Let $f(n)$ be defined for $n \in \mathbb{N}$ by $f(1) = 2$ and $f(n+1) = f(n)^2 - f(n) + 1$ for $n \geq 1$. Prove that for all $n > 1$
$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{f(1)} + \frac{1}{f(2)} + \cdots + \frac{1}{f(n)} < 1 - \frac{1}{2^{2^n}}.$$

Part 2

6. A sequence (x_n) is given by $x_1 = 2$ and $nx_n = 2(2n-1)x_{n-1}$ for $n > 1$. Prove that x_n is an integer for every $n \in \mathbb{N}$.

7. Let p, q, r be distinct real numbers that satisfy
$$q = p(4-p), \quad r = q(4-q), \quad p = r(4-r).$$
Find all possible values of $p + q + r$.

8. Prove that for every integer $n > 1$,
$$n\left((n+1)^{\frac{2}{n}} - 1\right) < \sum_{i=1}^{n} \frac{2i+1}{i^2} < n\left(1 - n^{-\frac{2}{n-1}}\right) + 4.$$

9. Suppose that w, a, b, c are distinct real numbers for which there exist real numbers x, y, z that satisfy the following equations:
$$\begin{array}{rcrcrcl} x &+& y &+& z &=& 1, \\ a^2x &+& b^2y &+& c^2z &=& w^2, \\ a^3x &+& b^3y &+& c^3z &=& w^3, \\ a^4x &+& b^4y &+& c^4z &=& w^4. \end{array}$$
Express w in terms of a, b, c.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

10. If a square is partitioned into n convex polygons, determine the maximum possible number of edges in the obtained figure.

    (You may wish to use the following theorem of Euler: If a polygon is partitioned into n polygons with v vertices and e edges in the resulting figure, then $v-e+n=1$.)

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com