6-th Irish Mathematical Olympiad 1993

May 8, 1993

Time: 3 hours each part.

Part 1

1. The real numbers α and β satisfy the equations

$$\alpha^3 - 3\alpha^2 + 5\alpha - 17 = 0,$$
$$\beta^3 - 3\beta^2 + 5\beta + 11 = 0.$$

   Compute $\alpha + \beta$.

2. A positive integer n is called *good* if it can be uniquely written simultaneously as $a_1 + a_2 + \cdots + a_k$ and as $a_1 a_2 \cdots a_k$, where a_i are positive integers and $k \geq 2$. (For example, 10 is good because $10 = 5+2+1+1+1 = 5 \cdot 2 \cdot 1 \cdot 1 \cdot 1$ is a unique expression of this form). Find, in terms of prime numbers, all good natural numbers.

3. A line l is tangent to a circle S at A. For any points B, C on l on opposite sides of A, let the other tangents from B and C to S intersect at a point P. If B, C vary on l so that the product $AB \cdot AC$ is constant, find the locus of P.

4. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ $(n \geq 1)$ be a polynomial with real coefficients such that $|f(0)| = f(1)$ and each root α of f is real and lies in the interval $[0, 1]$. Prove that the product of the roots does not exceed $1/2^n$.

5. For a complex number $z = x + iy$ we denote by $P(z)$ the corresponding point (x, y) in the plane. Suppose $z_1, z_2, z_3, z_4, z_5, \alpha$ are nonzero complex numbers such that

   (i) $P(z_1), \dots, P(z_5)$ are vertices of a complex pentagon Q containing the origin O in its interior, and

   (ii) $P(\alpha z_1), \dots, P(\alpha z_5)$ are all inside Q.

   If $\alpha = p + iq$ $(p, q \in \mathbb{R})$, prove that $p^2 + q^2 \leq 1$ and $p + q \tan \dfrac{\pi}{5} \leq 1$.

Part 2

6. Show that among any five points $P_1, dots, P_5$ with integer coordinates in the plane, there exists at least one pair (P_i, P_j) with $i \neq j$ such that the segment $P_i P_j$ contains a point Q with integer coordinates other than P_i, P_j.

7. Let a_i, b_i $(i = 1, \dots, n)$ be real numbers such that the a_i are distinct, and suppose that there is a real number α such that the product $(a_i + b_1)(a_i + b_2)\cdots(a_i + b_n)$ is equal to α for each i. Prove that there is a real number β such that $(a_1 + b_j)(a_2 + b_j)\cdots(a_n + b_j)$ is equal to β for each j.

8. If $1 \leq r \leq n$ are integers, prove the identity

$$\sum_{d=1}^{\infty} \binom{n-r+1}{d}\binom{r-1}{d-1} = \binom{n}{r}.$$

9. Prove that for every real number x with $0 < x < \pi$ and every natural number n

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots + \frac{\sin(2n-1)x}{2n-1} > 0.$$

10. (a) The rectangle $PQRS$ with $PQ = l$ and $QR = m$ ($l, m \in \mathbb{N}$) is divided into lm unit squares. Prove that the diagonal PR intersects exactly $l + m - d$ of these squares, where $d = (l, m)$.

    (b) A box with edge lengths $l, m, n \in \mathbb{N}$ is divided into lmn unit cubes. How many of the cubes does a main diagonal of the box intersect?

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