19-th Irish Mathematical Olympiad 2006

May 06, 2006

Time: 3 hours each part.

Part 1

1. Do there exist integers x, y, and z which satisfy the equation
$$z^2 = (x^2+1)(y^2-1)+n$$
when
   (a) $n = 2006$;
   (b) $n = 2007$?

2. P and Q are points on the equal sides AB and AC respectively of an isosceles triangle ABC such that $AP = CQ$. Moreover, neither P nor Q is a vertex of ABC . Prove that the circumcircle of the triangle APQ passes through the circumcenter of the triangle ABC.

3. Prove that a square of side 2.1 units can be completely covered by seven squares of side 1 unit.

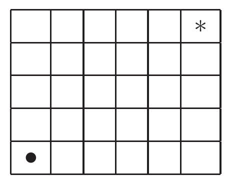
4. Find the greatest value and the least value of $x+y$, where x and y are real numbers, with $x \geq -2$, $y \geq -3$ and
$$x - 2\sqrt{x+2} = 2\sqrt{y+3} - y.$$

5. Determine, with proof, all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(1) = 1$, and
$$f(xy + f(x)) = xf(y) + f(x)$$
for all $x, y \in \mathbb{R}$.

www.imomath.com

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
Provided by Borislav Mirchev
www.imomath.com

6. The rooms of a building are arranged in a $m \times n$ rectangular grid (as shown below for the 5×6 case). Every room is connected by an open door to each adjacent room, but the only access to or from the building is by a door in the top right room. This door is locked with an elaborate system of mn keys, one of which is located in every room of the building. A person is in the bottom left room and can move from there to any adjacent room. However, as soon as the person leaves a room, all the doors of that room are instantly and automatically locked. Find, with proof, all m and n for which it is possible for the person to collect all the keys and escape the building.



$\bullet$ – starting position; $*$ - room with locked external door.

7. ABC is a triangle with points D, E on BC, with D nearer B; F, G on AC, with F nearer C; H, K on AB, with H nearer A. Suppose that $AH = AG = 1$, $BK = BD = 2$, $CE = CF = 4$, $\angle B = 60^\circ$ and that D, E, F, G, H, and K all lie on a circle. Find the radius of the incircle of the triangle ABC.

8. Suppose x and y are positive real numbers such that $x + 2y = 1$. Prove that
$$\frac{1}{x} + \frac{2}{y} \geq \frac{25}{1 + 48xy^2}.$$

9. Let n be a positive integer. Find the greatest common divisor of the numbers
$$\binom{2n}{1}, \binom{2n}{3}, \binom{2n}{5}, \dots, \binom{2n}{2n-1}.$$

10. Two positive integers n and k are given, with $n \geq 2$. In the plane there are n circles such that any two of them intersect at two points and all these intersection points are distinct. Each intersection point is colored with one of n given colors in such a way that all n colors are used. Moreover, on each circle there are precisely k di?erent colors present. Find all possible values for n and k for which such a coloring is possible.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
Provided by Borislav Mirchev
www.imomath.com