18-th Irish Mathematical Olympiad 2005

May 7, 2005

Time: 3 hours each part.

Part 1

1. Show that 2005^{2005} is a sum of two perfect squares, but not a sum of two perfect cubes.

2. Let D, E and F be points on the sides BC, CA and AB respectively of a triangle ABC, distinct from the vertices, such that AD, BE and CF meet at a point G. Prove that if the triangles AGE, CGD, BGF have equal area, then G is the centroid of $\triangle ABC$.

3. Prove that the sum of the lengths of the medians of a triangle is at least three quarters of its perimeter.

4. Determine the number of arrangements $a_1, a_2, \dots, a_{10}$ of the numbers $1, 2, \dots, 10$ such that $a_i > a_{2i}$ for $1 \le i \le 5$ and $a_i > a_{2i+1}$ for $1 \le i \le 4$.

5. Let a, b, c be nonnegative real numbers. Prove that
$$\frac{1}{3}\left((a-b)^2+(b-c)^2+(c-a)^2\right) \le a^2+b^2+c^2-3\sqrt[3]{a^2b^2c^2} \le (a-b)^2+(b-c)^2+(c-a)^2.$$

Part 2

6. Let X be a point on the side AB of a triangle ABC, different from A and B.Let P and Q be the incenters of the triangles ACX and BCX respectively, and let M be the midpoint of PQ. Prove that $MC > MX$.

7. Using the digits $1, 2, 3, 4, 5$, players A and B compose a 2005-digit number N by selecting one digit at a time: A selects the first digit, B the second, A the third and so on. Player A wins if and only if N is divisible by 9. Who will win if both players play as well as possible?

8. Let x be an integer and y, z, w be odd positive integers. Prove that 17 divides $x^{y^{z^w}} - x^{y^z}$.

9. Find the first digit to the left and the first digit to the right of the decimal point in the expansion of $\left(\sqrt{2}+\sqrt{5}\right)^{2000}$.

10. Suppose that m and n are odd integers such that m^2-n^2+1 divides n^2-1. Prove that m^2-n^2+1 is a perfect square.

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