16-th Irish Mathematical Olympiad 2003

May 10, 2003

Time: 3 hours each part.

Part 1

1. Find all solutions in (not necessarily positive) integers of the equation
$$(m^2+n)(m+n^2) = (m+n)^3.$$

2. P, Q, R, and S are (distinct) points on a circle. PS is a diameter and QR is parallel to the diameter PS. PR and QS meet at A. Let O be the center of the circle and let B be chosen so that the quadrilateral $POAB$ is a parallelogram. Prove that $BQ = BP$.

3. For each positive integer k¡ let a_k be the greatest integer not exceeding $\sqrt{k}$ and let b_k be the greatest integer not exceeding $\sqrt[3]{k}$. Calculate
$$\sum_{k=1}^{2003}(a_k - b_k).$$

4. Eight players, Ann, Bob, Con, Dot, Eve, Fay, Guy, and Hal compete in a chess tournament. No pair plays together more than once and there is no group of five people in which each one plays against all of the other four.
   (a) Write down an arrangement for a tournament of 24 games satisfying these conditions.
   (b) Show that it is impossible to have a tournament of more than 24 games satisfying these conditions.

5. Show that there is no function f defined on the set of positive real numbers such that
$$f(y) > (y-x)(f(x))^2$$
for all x, y with $y > x > 0$.

Part 2

6. Let T be a triangle of perimeter 2, and let a, b, and c be the lengths of the sides of T.
   (a) Show that
$$abc + \frac{28}{27} \geq ab + bc + ca.$$
   (b) Show that
$$ab + bc + ca \geq abc + 1.$$

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7. $ABCD$ is a quadrilateral. P is the foot of the perpendicular from D to AB, Q is the foot of the perpendicular from D to BC, R is the foot of the perpendicular from B to AD, and S is the foot of the perpendicular from B to CD. Suppose that $\angle PSR = \angle SPQ$. Prove that $PR = SQ$.

8. Find all solutions in integers x, y to the equation

$$y^2 + 2y = x^4 + 20x^3 + 104x^2 + 40x + 2003.$$

9. Let $a, b > 0$. Determine the largest number c such that

$$c \leq \max\left\{ax + \frac{1}{ax}, bx + \frac{1}{bx}\right\}$$

for all $x > 0$.

10. (a) In how many ways can 1003 distinct integers be chosen from the set $\{1, 2, \ldots, 2003\}$ so that no two of the chosen integers differ by 10?

(b) Show that there are $(3(5151) + 7(1700))101^7$ ways to choose 1002 distinct integers from the set $\{1, 2, \ldots, 2003\}$ so that no two of the chosen integers differ by 10.

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