13-th Irish Mathematical Olympiad 2000

May 6, 2000

Time: 3 hours each part.

Part 1

1. Consider the set S of all numbers of the form $a(n) = n^2 + n + 1$, $n \in \mathbb{N}$. Show that the product $a(n)a(n+1)$ is in S for all $n \in \mathbb{N}$ and give an example of two elements s,t of S such that $st \notin S$.

2. Let $ABCDE$ be a regular pentagon of side length 1. Let F be the midpoint of AB and let G and H be the points on sides CD and DE respectively $\angle GFD = \angle HFD = 30^\circ$. Show that the triangle GFH is equilateral. A square of side a is inscribed in $\triangle GFH$ with one side of the square along GH. Prove that
$$FG = t = \frac{2\cos 18^\circ \cos^2 36^\circ}{\cos 6^\circ} \quad \text{and} \quad a = \frac{t\sqrt{3}}{2+\sqrt{3}}.$$

3. Let $f(x) = 5x^{13} + 13x^5 + 9ax$. Find the least positive integer a such that 65 divides $f(x)$ for every integer x.

4. The sequence $a_1 < a_2 < \cdots < a_M$ of real numbers is called a *weak arithmetic progression* of length M if there exists an arithmetic progression $x_0, x_1, \dots, x_M$ such that
$$x_0 \le a_1 < x_1 \le a_2 < x_2 \le \cdots \le a_M < x_M.$$
   (a) Prove that if $a_1 < a_2 < a_3$ then (a_1, a_2, a_3) is a weak arithmetic progression.
   (b) Prove that any subset of $\{0, 1, 2, \dots, 999\}$ with at least 730 elements contains a weak arithmetic progression of length 10.

5. Consider all parabolas of the form $y = x^2 + 2px + q$ for $p, q \in \mathbb{R}$ which intersect the coordinate axes in three distinct points. For such p,q, denote by $C_{p,q}$ the circle through these three intersection points. Prove that all circles $C_{p,q}$ have a point in common.

Part 2

6. Prove that if x,y are nonnegative real numbers with $x+y=2$, then
$$x^2y^2(x^2+y^2) \le 2.$$

7. In a cyclic quadrilateral $ABCD$, a,b,c,d are its side lengths, Q its area, and R its circumradius. Prove that
$$R^2 = \frac{(ab+cd)(ac+bd)(ad+bc)}{16Q^2}.$$
   Deduce that $R \ge \dfrac{(abcd)^{3/4}}{Q\sqrt{2}}$ with equality if and only if $ABCD$ is a square.

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8. For each positive integer n find all positive integers m for which there exist positive integers $x_1 < x_2 < \cdots < x_n$ with

$$\frac{1}{x_1} + \frac{2}{x_2} + \cdots + \frac{n}{x_n} = m.$$

9. Show that in each set of ten consecutive integers there is one that is coprime with each of the other integers. (For example, in the set $\{114, 115, \ldots, 123\}$ there are two such numbers: 119 and 121.)

10. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with nonnegative real coefficients. Suppose that $p(4) = 2$ and $p(16) = 8$. Prove that $p(8) \leq 4$ and find all such p with $p(8) = 4$.

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