Indian IMO Team Selection Tests 1997

First Practice Test
Mumbai

1. The line l intersects the boundary of a triangle ABC in two different points. Prove that any two of the following statements imply the third:

   (i) l bisects the perimeter of $\triangle ABC$;

   (ii) l bisects the area of $\triangle ABC$;

   (iii) l passes through the incenter of $\triangle ABC$.

2. The sequence $(a_n)_{n=1}^{\infty}$ of positive integers is defined by $\sum_{d|n} a_d = 2^n$ for every $n \in \mathbb{N}$. Prove that:

   (a) $pq \mid a_{pq}$ for any distinct primes p, q

   (b) $p^m \mid a_{p^m}$ for any prime p and natural number m.

3. If $a_1, a_2, \dots, a_n$ $(n > 2)$ are odd positive integers whose mutual differences are all distinct, prove that
$$\sum_{i=1}^{n} a_i > \frac{n(n^2+3)}{4}.$$

4. In a tournament with $n \geq 2$ players, every two play exactly one match and there are no draws. Show that there is a player X such that, for every other player Y, either X defeated Y or X defeated a player who in turn defeated Y.

Second Practice Test
Mumbai, May 13

1. Let ABC be a triangle of area S. Show that there exists a line l in the plane of the triangle such that the area common to $\triangle ABC$ and its reflection in line l is larger than $2S/3$.

2. A divisor $d > 0$ of a positive integer n is said to be a *unitary divisor* if $\gcd(d, \frac{n}{d}) = 1$. (For example, the unitary divisors of 12 are $1, 3, 4, 12$). Prove that if the sum of all unitary divisors of n equals $2n$ then n cannot be odd.

3. If x, y are nonnegative real numbers satisfying the equation
$$2x + y + \sqrt{2xy + 3x^2 + y^2} = 5,$$
prove that $xy^2 < 1$.

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D. Djukić, V. Janković, I. Matić, N. Petrović
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4. Let $\{a_i\}$ and $\{b_i\}$ ($i = 1, 2, \dots, n$) be two distinct collections of nonnegative integers (with possible repetitions in each collection). Suppose that the two collections $\{a_i + a_j\}$ and $\{b_i + b_j\}$ ($1 \le i < j \le n$) are identical. (For example, this holds for the 4-element collections $\{0, 2, 2, 2\}$ and $\{1, 1, 1, 3\}$.) Prove that n is a power of 2.

First Test

Mumbai, May 15

1. Let I be the incenter of triangle ABC and D, E be the midpoints of the respective sides AC, AB. Ray DI meets AB at P and ray EI meets AC at Q. Show that $AP \cdot AQ = AB \cdot AC$ if and only if $A = 60^\circ$.
2. Find all pairs of prime numbers (p, q) for which pq divides $2^p + 2^q$.
3. Suppose that the equation $z^{k_1} + z^{k_2} + \cdots + z^{k_r} = 0$ has a complex root z on the unit circle, where $0 \le k_1 < k_2 < \cdots < k_r$ are integers. Show that z is a root of unity when $r = 2, 3, 4$.
4. Let X be the set of positive integers $n \le 1997$ which are not powers of 2, and let A be a 997-element subset of X. Show that there are two integers $x, y \in A$ such that $x + y$ is a power of 2.

Second Test

Mumbai, May 19

1. In a quadrilateral $ABCD$, points P, Q, R, S are the midpoints of sides AB, BC, CD, DA, respectively. The lines AB and DC intersect in X and the lines AD and BC intersect in Y. Prove that the orthocenters of triangles XRP and YSQ coincide if and only if $ABCD$ is a cyclic quadrilateral.
2. Let a and b be two coprime positive integers with $a + b$ odd. The set S of positive integers satisfies the following conditions:

   (i) $a, b \in S$;

   (ii) $x + y + z \in S$ whenever $x, y, z \in S$.

   Show that every integer $n \ge 2ab$ is in S.
3. If a, b, c are nonnegative real numbers with $a + b + c = 1$, prove the inequality
$$\frac{a}{1+bc} + \frac{b}{1+ca} + \frac{c}{1+ab} \ge \frac{9}{10}.$$
4. A sequence of positive integers $a_1, a_2, \dots, a_{1995}$ with the sum 3989 is given. Show that there is a block of r successive a_i's ($r \ge 1$) whose sum is 95.

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Third Test

Mumbai, May 21

1. Let E and F be points on sides BC, CD respectively of a square $ABCD$ and let diagonal BD meet AE at P and AF at Q. Prove that if $BE \neq DF$ and $BP \cdot CE = DQ \cdot CF$, then the points P, Q, F, E, C lie on a circle.

2. The function $f : \mathbb{N} \to \mathbb{N}$ is defined by $f(n) = n + [\sqrt{n}]$. Prove that for any positive integer m the sequence m, $f(m)$, $f(f(m))$, $f(f(f(m))), \dots$ contains infinitely many squares.

3. Suppose that x and y are different real numbers such that
$$Q_n = \frac{x^n - y^n}{x - y}$$
is an integer for some four consecutive positive integers n. Prove that Q_n is an integer for all positive integers n.

4. Find all permutations $(a_1, a_2, \dots, a_n)$ of $1, 2, \dots, n$ that satisfy
$$a_1 + 1 \leq a_2 + 2 \leq \cdots \leq a_n + n.$$

Fourth Test

Mumbai, May 24

1. Let G be the centroid of triangle ABC. The rays AG, BG, CG meet the circumcircle of triangle ABC at points $\hat{A}, \hat{B}, \hat{C}$. If $\angle A > \angle B > \angle C$, prove that the largest angle of triangle $\hat{A}\hat{B}\hat{C}$ is at vertex $\hat{B}$.

2. Show that any integer divisible by 3 can be written as a sum of four cubes. (For example, $6 = 2^3 + (-1)^3 + (-1)^3 + 0^3$.)

3. Given n complex numbers $x_1, x_2, \dots, x_n$, define
$$y_j = \sum_{k=1}^{n} x_k x_{j-k} \quad \text{for } 1 \leq j \leq n,$$
where the indices are taken modulo n. Prove that if $y_j = 0$ for all j, then $x_k = 0$ for all k.

4. A sequence $a_0, a_1, \dots, a_{n-1}$ of 0's and 1's of length n is called *very odd* if $\sum_{i=0}^{n-k-1} a_i a_{i+k}$ is an odd number for $k = 0, 1, \dots, n-1$. Prove that if there is a very odd sequence of length n then n is of the form $4k$ or $4k+1$ ($k \in \mathbb{N}_0$).

Fifth Test

Mumbai, May 28

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1. Let ABC be a triangle in the coordinate plane whose all vertices are lattice points. Suppose that there is a unique lattice point G in the interior of the triangle and that there are no lattice points on the sides of the triangle other than the vertices. Prove that G is the centroid of $\triangle ABC$.

2. If a,b,c are positive numbers, prove the inequality

$$\frac{1}{a(b+1)}+\frac{1}{b(c+1)}+\frac{1}{c(a+1)} \geq \frac{3}{abc+1}\,.$$

3. (a) Given natural numbers $1=r_1<r_2<\cdots<r_k \leq n$, find the number of partitions of set $X=\{1,2,\ldots,n\}$ into k subsets $A_1,\ldots,A_k$ such that the least element in A_j is r_j for $1 \leq j \leq k$.

   (b) Prove that $S(n,k) \geq \binom{n-j-1}{n-k}^2$, where $S(n,k)$ is the number of partitions of X into k subsets and $j=[k/2]$. (The $S(n,k)$ are called the Stirling numbers of the second kind).

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